Cahn-Hilliard-Navier-Stokes Turbulence: Physics, Numerical Simulations, and Mathematics

Rahul Pandit
Indian Institute of Science, India

I introduce the Cahn-Hilliard-Navier-Stokes (CHNS) equations and show that they provide a natural theoretical framework for describing two-phase flows, both laminar and turbulent. I then cover some of our recent results, based on extensive numerical simulations, of turbulence in the CHNS system and the important dimensionless control parameters for such turbulence. In particular, I discuss (a) the multifractal fluctuations of a droplet in a two-phase, turbulent flow and (b) the suppression of phase separation by turbulence in a two-fluid system. I end with a Beale-Kato-Majda-type theorem for the regularity of solutions of the three-dimensional CHNS equations. The first two parts of this work have been done with Nairita Pal and Prasad Perlekar; the last part has been done with Nairita Pal, Anupam Gupta, and John Gibbon

Energy dissipation in blood flows related to cardiovascular problems

Hiroshi Suito[1], Viet Q.H. Huynh[1], Kenji Takizawa[2] and Takuya Ueda[3]

[1]Advanced Institute for Materials Research, Tohoku University, Katahira 2-1-1, Aoba-ku, Sendai, 980-8577, Japan,
Email: hiroshi.suito@tohoku.ac.jp
[2]Faculty of Science and Engineering, Waseda University, Japan
[3]Department of Radiology, Tohoku University Hospital, Japan

Abstract

We present numerical experiments related to cardiovascular problems such as the aortic aneurysm, which is one of the life-threatening diseases. A number of patient-specific models of the aorta as constructed from CT scans are considered. Blood flows are affected by curvature and torsion distributions as well as branches to smaller arteries. Such geometrical characteristics of the vessels bring about differences in the flow characteristics, which lead to different energy dissipation patterns and outcomes.

This work was supported by JST CREST Grant Number JPMJCR15D1, Japan.

Friction induced instability of surface of velocity discontinuity of a shallow-water flow

Yasuhide Fukumoto[1], Thi Thai LE[2], and Liangbing JIN[3]
[1] *Institute of Mathematics for Industry, Kyushu University, Japan,*
[2] *Graduate School of Mathematics, Kyushu University, Japan*
[3] *Zhejiang Normal University, China*

We examine the frictional effect on the stability of velocity discontinuity in a shallow-water stream. The fluid is moving with a uniform velocity U in a half region but is at rest in the other, and the bottom surface is assumed to exert the drag force, quadratic in velocity, on the thin fluid layer. In the absence of the drag, the interface of velocity discontinuity undergoes the instability of the Kelvin-Helmholtz type for $U \leq \sqrt{8}c$, but is stabilized for $U > \sqrt{8}c$, with c being the propagating speed of the gravity wave. The bottom drag, however small it may be, destabilizes the interface in the latter range of the Froude number U/c. For small values of the drag coefficient, the growth rate is written down in tidy form. We find by an asymptotoc analysis that the instability never ceases as the drag strength is increased.

Numerical implementation of the construction of dissipative Euler solutions with the Mikado flow

Takeshi Matsumoto
Department of Physics, Kyoto University, Japan

Certain weak solutions of the incompressible Euler equations which dissipate the energy are likely to follow the same laws of high Reynolds number turbulent flows. A mathematical method to construct such a weak solution has been proposed by De Lellis, Szekelyhidi and coworkers. In this talk, we explain why these weak solutions are appealing also for physicists and mention our current attempt at numerical implementation of their mathematical construction.

Study of Biot-Savart models for vortex reconnection

Yoshifumi Kimura
Graduate School of Mathematics, Nagoya University, Japan

Vortex reconnection under Biot-Savart evolution is investigated geometrically and numerically using vortex filament models. It is shown that the tips of these vortices approach each other accelerating as they do so to form a finite-time singularity. The minimum separation of the vortices and the maximum velocity and axial strain rate exhibit nearly self-similar Leray scaling. By way of validation of the models, the structure of the eigenvalues and eigenvectors of the rate-of-strain tensor is investigated. At the tips, it is observed that the second eigenvalue is positive and the corresponding eigenvector is tangent to the filament, implying persistent stretching of the vortex. This is a joint work with H. K. Moffatt (Cambridge University).

On scalar and vectorial spherical harmonics for analysing and calculating strongly anisotropic shear-driven rotating turbulent flows

Claude Cambon
Ecole Centrale Lyon, France

Isotropy is often recovered in shear-driven and/or in thermally-driven turbulent flows for length scales significantly smaller than typical threshold scales (Corrsin 1958, Ozmidov, Woods-Hopfinger-Zeman (1984)); weakly anisotropic theories and models were proposed for length scales close to, or smaller than, these threshold ones. On the other hand, an extended scale-by-scale analysis reveals that anisotropy is strong in the largest scales and cannot be represented by angular harmonic expansions at low degree. Spherical harmonics are used for representing the three-dimensional spectral distribution of kinetic energy, with directional anisotropy, and then polarization anisotropy, that ought to be disentangled (e.g. CC and Rubinstein 2006, Rubinstein *et al.* 2015). On this occasion, a large historical review of using both scalar and vectorial harmonics is proposed. Regarding realizations of incompressible velocity fields, various toroidal/poloidal decompositions are discussed and reconciled or not: Chandrasekkar 1963, Craya (1957), Herring (1964), Riley (1981), CC and Teissèdre (1985), Rieutord (1987). Linkage to symmetry groups, as SO(3), is touched upon, as well as Wiener-Kinchin relationship for passing from spectra or co-spectra to structure functions in physical space , and conversely. Finally, applications are illustrated by the development of two-point statistics of homogeneous turbulence in the presence of uniform mean shear and system rotation. The so-called 'rapid distortion' limit ought to be captured in a large range of scales, even if non-linearity is significant. For instance, this limit is relevant in the infrared domain, in which purely linear dynamics compete with the backscatter transfer, that is mediated by non-linear non-local interactions from small scales. Depending on various dynamics, from algebraic growth to exponential growth or decay, spherical harmonics are used either for data processing only, or as a basis for projecting dynamical equations. In our most elaborate model-equations, exact linear effects that reflect rapid distortion, if considered alone, are solved with maximum accuracy on spherical shells with many angles. Meanwhile the nonlinear terms are closed by a generalized EDQNM (Orszag 1970, Mons *et al.* 2016) procedure. Cross-validation with DNS in distorted boxes (Rogallo 1981, Lesur 2005, "snoopy" code) are expected.

WEAKENED REGULARITY CONDITIONS TO INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

HI JUN CHOE

1. Introduction

Incompressible viscous fluid is governed by incompressible Navier-Stokes equations:

$$u_t - \nu \Delta u + \nabla \cdot (u \otimes u) + \nabla p = 0$$

$$div(u) = 0,$$

where ν is viscosit constant, $u(x,t)$ is fluid velocity field at (t,x) and $p(x,t)$ is pressure at (x,t). For convenience of presentation we assume viscosity is one.

In 1934, Leray proved that there is a weak solution in $L^{2,\infty}$ for initial value problem when the initial data $u(x,0) \in L^2$ and weaklly solenoidal although the uniqueness is still unkown. He constructed weak solution by Galerkin method. The energy inequality

$$sup_t ||u||_{L^2}^2(t) + \int_0^\infty \int ||\nabla u|^2 dxdt \leq ||u(x,0)||_{L^2}^2$$

implies strong compactness in $L^{2,2}$ and one obtain strong convergece of nonlinear term $u \otimes u$ which is convection. We can not mention all important results related to exsitence questions in various circumstances. With perturbation for nonlinear term under boundedness assumption for the velocity u, we obtain a stability estimate and therefore the uniqueness follows. Similarly we have a stability estimate if the boundedness of enstrophy holds so that

$$\sup_t \int |\nabla u|^2 dx < \infty.$$

We note that from solenodality condition the vorticity ω satisfies

$$\int |\nabla u|^2 dx = \int |\omega|^2 dx, \quad \omega = \nabla \times u.$$

We call a solution strong if it is weak and has bounded enstrophy.

Among the Galilean transforms, we have a special scaling for Navier-Stokes equations:

$$(u_\lambda(x,t), p_\lambda(x,t)) = (\lambda u(\lambda x, \lambda^2 t), \lambda^2 p(\lambda x, \lambda^2 t)),$$

since (u_λ, p_λ) is a solution, too. First we find that the Lebesgue spaces

$$L^{p,q}, \quad \frac{3}{p} + \frac{2}{q} = 1, \quad p > 3, \quad q > 2$$

are invariant under the scaling and the condition of parameters is called Ladyzhenskay-Prodi-Serrin condition. There are many authors, including Serrin, Prodi, Kato, Fujita, Kozono and Kang, who considered regularity property. One notable result in view of scaling is due to Kato.

Theorem 1.1. *(**Kato**) Suppose $u(x,0)$ is in L^3, then there is a positive time T such that a strong soution exists in $(0,T)$.*

We skip recent activities to extend short time existence of strong solutions for various scale invariant spaces.

In theories of minimal surface, harmonic map, elliptic system and freeboundary, ϵ-regularity argument are very successful by scale invariance and perturbation of linear operator. In 1970's, Scheffer was the first person who considered local regularity condition in the theory of ϵ-regularity. His parabolic dimension estimate of possible singular set in space-time was two. By the way, Leray already showed the Hausdorff dimension of singular time is less than $\frac{1}{2}$ and it is still unkown that $\frac{1}{2}$ is best or not. We like to emphasize that there are singular solutions to minimal surface, harmonic map, elliptic system and freeboundary although one might need higher space dimension to construct. In Navier-Stokes equations, singular set S is the set of point z in domain such that u is unbounded in any neighborhood of z and the space dimension is strictly physical dimension 3.

The Galerkin weak solution due to Leray for whole domian and Hoff for bounded domain is not convenient to consider local properties and Scheffer introduced the suitable weak solution.

Definition 1.2. *We say $(u,p) \in V \times L^{3/2}(\Omega_T)$ is suitable weak solution to the initial boundary value problem if for all $\phi \in C_0^\infty(R^3_+ \times R_+)$*

$$\int u \cdot \phi_t dz + \int \nabla u : \nabla \phi dz + \int u \otimes u : \nabla \phi dz - \int p div\phi dz = 0 \tag{1.1}$$

and u is weakly divergence free for almost all time, satisfies the localized energy inequality for almost all t

$$\begin{aligned} \int |u(x,t)|^2 \phi dx + 2\int_0^t \int |\nabla u|^2 \phi dx ds \\ \leq \int_0^t \int |u|^2(\phi_t + \Delta\phi) dx ds + \int_0^t \int (|u|^2 + 2p) u \cdot \phi dx ds \end{aligned} \tag{1.2}$$

for all nonnegative $\phi \in C_0^\infty(R^3 \times R_+)$ and

$$\int |u(x,t) - u_0(x)|^2 dx \to 0 \quad as \quad t \to 0,$$

where the initial data u_0 is weakly divergence free in $L^2(R^3_+)$.

The following existence theorem is due to Scheffer and Caffarelli-Kohn-Nirenberg.

Theorem 1.3. *There is a suitable weak solution to the initial boundary value problem.*

We define a ball $B_r(x_0) = \{x : |x - x_0| < r\}$ and a parabolic cylinder $Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r_0^2)$. The following ϵ regularity theorem is due to Caffarelli-Kohn-Nirenberg:

Theorem 1.4. *There is an absolute constant ϵ_0 such that*

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r} |\nabla u|^2 dz \le \epsilon_0$$

implies that for an r_0

$$\sup_{z\in Q_{r_0}} |u(z)| \le \frac{c}{r_0}$$

for an absolute constant c.

As a natural consequence we have an estimate of Hausdorff dimension of singular set S.

Theorem 1.5. *We let $L(\delta)$ be the family of all coverings $\{Q_{r_i}(z_i), r_i < \delta\}$ of S and $h(r) = r|logr|$. We define*

$$\Psi_\delta(S, h) = \inf_{L(\delta)} \sum_i h(r_i)$$

and set

$$\Lambda(S, h) = \lim_{\delta\to 0} \Psi_\delta(E, h).$$

Then

$$\Lambda(S, h) = 0.$$

Recently we observed that the possible oscillation of local energy density is large near singular point. To see that we define

$$E_q(z, r) = r^{-5+2q} \iint_{Q(z,r)} |\nabla u|^q dxdt$$

and denote

$$\overline{E}_q(z) = \limsup_{r\to 0} E_q(z, r) \quad \text{and} \quad \underline{E}_q(z) = \liminf_{r\to 0} E_q(z, r).$$

Theorem 1.6. *Let $9/5 \le q < 2$. There exists a positive number ϵ such that $z \in \Omega \times (0, T)$ is a regular point if*

$$\overline{E}_q(z)^{(5-q)/(q-1)} \underline{E}_q(z) < \epsilon.$$

As in the case of Kato, Escauriaza–Sergin–Šverák proved that if $u \in L^{3,\infty}$ then it is regular. From the scale invariant property of Navier-Stokes equations, we have a natural question of type one singularity and the smallest space for the type one singularity is $weak - L^3$ space in space. We study local regularity properties of a $L^\infty(0,T; weak - L^3(\mathbb{R}^3))$ solution u to the Cauchy problem of the incompressible Navier–Stokes equations, where $u \in weak - L^3$ if

$$\sup_h h^3 |\{u > h\}| < M$$

for some M.

As an application, we conclude that there are at most a finite number of blowup points at any singular time t. The condition that the weak Lebesgue space norm of the veclocity field u is bounded in time is encompassing type I singularity and significantly weaker than the end point case of the so-called Ladyzhenskaya–Prodi–Serrin condition proved by Escauriaza–Sergin–Šverák.

Theorem 1.7. *For each $M > 0$ there exists a positive number $\epsilon(M) < 1/4$ such that if a weak solution $u \in V(Q_T)$ to the Cauchy problem satisfies the conditions*

$$ess \sup_{0 \le t \le T} ||u||_{weak-L^3} \le M \tag{1.3}$$

and for some $z_0 = (x_0, t_0) \in Q_T$ and $0 < r \le \sqrt{t_0}$

$$\frac{1}{r^3} |\{x \in B(x_0, r) : |u(x, t_0)| > \frac{\epsilon}{r}\}| \le \epsilon,$$

where $|E|$ denotes the Lebesgue measure of the set E, then u is bounded in the space-time cylinder $Q(z_0, \epsilon r)$.

As an application of this criterion, we are able to estimate the size of possible singular points at a singular time t, denoted by

$$\Sigma(t) = \{x : (x, t) \in \Sigma\}.$$

We know that the Hausdorff dimension of the possible singular time is at most $1/2$. Many researchers have been investigating the size of $\Sigma(t)$ at the singular time t under various conditions on u. Seregin obtained a result on estimating $\Sigma(t)$ under the slightly weaker condition. More recently, Wang–Zhang gave a unifying results on the number of singular points under the Ladyzhenskaya–Prodi–Serrin type conditions.

Utilizing Theorem 1.7, we can obtain the following theorem which shows that the number of possible singular points at any singular time t is at most finite.

Theorem 1.8. *Suppose $u \in V_\sigma^2(Q_T)$ is a weak solution to the Cauchy problem and satisfies the condition* (1.3) *for some $M > 0$. Then there exist at most finite number $N(M)$ of singular points at any singular time t.*

At each singular time t, only a few singular points exist, yet we do not know that blowup points are of type I or not.

References

[1] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, Comm. Pure Appl. Math. *35*(1982), 771-831.

[2] D. Chae and J. Wolf. On the liouville theorem for self similar solutions to the Navier-Stokes equations. *Preprint*, 2016.

[3] H. J. Choe, Partial Regularity of Leray-Hopf solution for the Navier-Stokes Equations, submitted.

[4] H. J. Choe, Boundary regularity of weak solutions of the Navier-Stokes equations, J. Differential Equations *149* (1998), no. 2, 211-247.

[5] H. J. Choe and J. Lewis, On the singular set in the Navier-Stokes equations, J. Functional Analysis *175*(2000), 348-369.

[6] L. Escauriaza, G. Seregin, and V. Šverák. Backward uniqueness for parabolic equations. *Arch. Ration. Mech. Anal.*, 169(2):147–157, 2003.

[7] G. Galdi, C. Simader and H. Sohr, *On the Stokes problem in Lipshitz domain,* Annali di Mat. Pura Appl., (IV), 167 (1994), pp. 147-163.

[8] E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr. *4*(1951), 213-231.

[9] S. Gustafson, K. Kang, and T. P. Tsai. Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations. *Comm. Math. Phys.*, 273(1):161–176, 2007.

[10] E. Hopf. Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen. *Math. Nachr.*, 4:213–231, 1951.

[11] J. Kinnunen and J. Lewis, Higher integrability for parabolic systems of p-Laplacian type, Duke Math. J. *102*(2000), 253-271.

[12] H. Kozono. Removable singularities of weak solutions to the Navier-Stokes equations. *Comm. Partial Differential Equations*, 23:949–966, 1998.

[13] O. A. Ladyženskaja. Uniqueness and smoothness of generalized solutions of Navier-Stokes equations. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 5:169–185, 1967.

[14] O. A. Ladyzhenskaya and G. A. Seregin. On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations. *J. Math. Fluid Mech.*, 1(4):356–387, 1999.

[15] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta. Math. *63*(1934) 193-248.

[16] F. Lin, A new proof of the Caffarelli, Kohn and Nirenberg theorem, Comm. Pure Appl. Math. *51*(1998), 241-257.

[17] J. Neustupa. Partial regularity of weak solutions to the Navier-Stokes equations in the class $L^\infty(0,T;L^3(\Omega)^3)$. *J. Math. Fluid Mech.*, 1(4):309–325, 1999.

[18] G. Prodi. Un teorema di unicità per le equazioni di Navier-Stokes. *Ann. Mat. Pura Appl. (4)*, 48:173–182, 1959.

[19] V. Scheffer, Hausdorff measure and the Navier-Stokes equations, Comm. Math. Phys. *55*(1977), 97-112

[20] V. Scheffer, Boundary regularity for the Navier-Stokes equations. Comm. Math. Phys, *85* (1982), no. 2, 275-299.

[21] G. A. Seregin, Local regularity of suitable weak solutions to the Navier-Stokes equations near the boundary, J. Math. Fluid Mech. *4* (2002), no. 1, 1-29.

[22] J. Serrin. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962)*, pages 69–98. Univ. of Wisconsin Press, Madison, Wis., 1963.

[23] V. A. Solonnikov, Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator. (Russian) Uspekhi Mat. Nauk 58 (2003), no. 2(350), 123–156; translation in Russian Math. Surveys 58 (2003), no. 2, 331-365

[24] E. Stein, "Singular Integrals and Differentiability Properties of Functions", Princeton Univ. Press, NJ, 1970.

[25] R. Temam, "Navier-Stokes Equations, Theory and Numerical Analysis," North-Holland, Amsterdam/New York, 1977.

Hi Jun Choe: CMAC/Department of Mathematics, Yonsei University, 50 Yonsei-ro, Seodaemun-gu, Seoul, South Korea 120-749

E-mail address: choe@yonsei.ac.kr

Recent works on the Naiver-Stokes equations with harmonic analysis

Okihiro Sawada
Gifu University, Japan

Bourgain-Pavlovic proved the ill-posedness theory of 3D Navier-Stokes equations in critical Besov spaces by norm-inflation argument. The proof is based on tools of the harmonic analysis. However, seeking the structure of velocity field, a blow-up does not occur. The key is the annihilation of gradient of pressure terms. This fact implies that the maximum principle is applicable, and then global-in-time smooth unique solutions exist. The same fact is obtained on the Euler equations. Some recent ill-posedness results by Bourgain-Li and Tao are also discussed.

Time decay estimate with diffusive property for solution to the compressible Navier-Stokes system

Takayuki Kobayashi
Osaka University, Japan

Time decay estimate of a solution to the compressible Navier-Stokes-Korteweg system is studied. It is known that the system describes two phase flow with phase transition between liquid and vapor as a diffuse interface model in a compressible fluid. Concerning time decay estimate, Wang and Tan ([3]) show convergence rates of L2 norm of a solution for 3 dimensional case under small initial value around constant state. The decay rate is same as that of heat kernel. In this talk concerning the linearized problem, the decay estimates with diffusive wave property (Cf., [1, 2]) for initial date are derived. Furthermore, as an application, we give time decay estimate of a solution to nonlinear system. In contrast to the compressible Navier-Stokes system, for linear system regularities of initial dates are lower and independent of the order of derivative of the solution owing to smoothing effect from the Korteweg tensor. Furthermore, for the nonlinear system diffusive wave properties are obtained with initial dates having lower regularity than that of studies of the compressible Navier-Stokes system and [3]. The results in this talk were obtained in a joint work with K. Tsuda (Osaka University, Japan).

References

[1] D. Hoff and K. Zumbrun, Multi-dimensional diffusion waves for the Navier- Stokes equations of compressible flow, Indiana Univ. Math. J., 44, (1995), pp.603-676.
[2] T. Kobayashi and Y. Shibata, Remark on the rate of decay of solutions to linearized compressible Navier-Stokes equations, Pacific Journal of Mathematics, 207 (2002), pp. 199-234.
[3] Y. Wang and Z. Tan, Optimal decay rates for the compressible fluid models of Korteweg type, J. Math. Anal. Appl., 379 (2011), pp. 256-271.

GLOBAL WELL-POSEDNESS OF THE TWO-DIMENSIONAL EXTERIOR NAVIER-STOKES EQUATIONS FOR NON-DECAYING DATA

K. ABE

Abstract. We consider the two-dimensional Navier-Stokes equations in an exterior domain $\Omega \subset \mathbb{R}^2$:

$$\begin{aligned} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 \quad &\text{in} \quad \Omega \times (0, \infty), \\ \operatorname{div} u = 0 \quad &\text{in} \quad \Omega \times (0, \infty), \\ u = 0 \quad &\text{on} \quad \partial\Omega \times (0, \infty), \\ u = u_0 \quad &\text{on} \quad \Omega \times \{t = 0\}. \end{aligned}$$

It is well known that the two-dimensional Navier-Stokes equations is globally well-posed for initial data with finite energy (Leray 1934, Ladyzhenskaya 1959). However, global well-posedness is unknown in general for initial data with infinite energy. In this talk, we report global well-posedness of the problem for bounded initial data with a finite Dirichlet integral and unique existence of asymptotically constant solutions for arbitrary large Reynolds numbers.

Viscous shock wave and singular limit for hyperbolic systems with Cattaneo's law

Tohru Nakamura
Kumamoto University, Japan

In this talk, we consider large time behavior of solutions to scalar conservation laws with an artificial heat flux term. In the case where the heat flux is governed by Fourier's law, the equation is scalar viscous conservation laws. In this case, existence and asymptotic stability of one-dimensional viscous shock waves have been studied. The main concern in the current talk is a 2×2 system of hyperbolic equations with relaxation which is derived by prescribing Cattaneo's law for the heat flux. We consider the one-dimensional Cauchy problem for the system of Cattaneo-type and show existence and asymptotic stability of viscous shock waves. We also obtain the convergence rate by utilizing the weighted energy method. By letting the relaxation time zero in the system of Cattaneo-type, the system is formally deduced to scalar viscous conservation laws of Fourier-type. This is a singular limit problem which occurs an initial layer. We also consider the singular limit problem associated with viscous shock waves.

Spectrum of turbulence in nonlinear Schrödinger (Gross-Pitaevskii) equation

Kyo Yoshida[1], Yoshiyuki Tsuji[2], and Hideaki Miura[3]
[1] *Faculty of Pure and Applied Sciences, University of Tsukuba,*
[2] *Graduate School of Engineering, Nagoya University,*
[3] *National Institute for Fusion Science*

Dynamical properties of quantum fluids such as superfluid ^{4}He or Bose-Einstein condensates (BECs) of ultracold atoms are described by the order parameter field $\psi(\boldsymbol{x}, t) \in \mathbb{C}$, where $\boldsymbol{x}$ and t are space and time coordinates. Under certain conditions, the dynamics of $\psi(\boldsymbol{x}, t)$ is governed by the nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi - \mu\psi + g|\psi|^2\psi,$$

which is also called the Gross-Pitaevskii equation. Here, m is the mass of the particles, μ the chemical potential, g the coupling constant and $\hbar$ is the reduced Planck constant.

Turbulent states of quantum fluids have received considerable attention both in experimental and numerical studies. See, e.g. Ref. [1] for a review for the numerical studies. Due to the recent progress in the control of ultracold atomic gas, it has become possible to stir BEC of ^{87}Rb and generate turbulent states [2,3]. Data of the spectrum $F(k) \propto \int \mathrm{d}\boldsymbol{k}'\ \delta(|\boldsymbol{k}'| - k)\langle|\psi_{\boldsymbol{k}'}|^2\rangle$, where $\psi_{\boldsymbol{k}}$ is the Fourier transform of $\psi(\boldsymbol{x})$, $\delta(k)$ is the Dirac delta function and $\langle\,\cdot\,\rangle$ denotes an ensemble average, have been accumulated both in experiments and numerical simulations. However, the synthetic understanding of the spectrum has not been established, especially in the case such that the nonlinearity in the NLS equation is strong.

In the present study, we performed the numerical simulation of NLS in a three-dimensional box with periodic boundary conditions in each coordinate direction. Pumping and dissipation of the mass are applied in small- and large-wavenumber regions, respectively, in order to maintain a turbulent state. For some cases such that the nonlinearity is strong, we observed $F(k) \propto k^{-2}$ which is consistent with the theoretical prediction for the energy-transfer region in Ref. [4].

References

[1] M. Tsubota, K. Fujimoto, and S. Yui. J. Low Temp. Phys. **188**(2017) 119.
[2] N. Navon *et al.*. Nature **539**(2016) 72.
[3] M. C. Tsatsos *et al.*. Phys. Rep. **622**(2016) 1.
[4] K. Yoshida and T. Arimitsu. J. Phys. A: Math. Theor. **46**(2013) 335501.

Self-similarity of heavy-particle clustering in the inertial range of turbulence

Taketo Ariki[1], Kyo Yoshida[2], Keigo Matsuda[3], & Katsunori Yoshimatsu[1]

[1] *Institute of Materials and Systems for Sustainability, Nagoya University*

[2] *Division of Physics, Faculty of Pure and Applied Sciences, University of Tsukuba*

[3] *Center for Earth Information Science and Technology, Japan Agency for Marine-Earth Science and Technology*

Inertial particles suspended in fluid turbulence represent spatially non-uniform distribution due to their non-negligible inertia, forming clusters of variable sizes. In the inertial range of homogeneous isotropic turbulence, we found a power-law behavior in particle's statistics which indicates self-similarity of particle clustering.

We consider particle's number-density field n and its normalized fluctuation, i.e. $\theta(\equiv n/\langle n\rangle - 1)$. A perturbation analysis and Kolmogorov's inertial-range scaling result in a self-similar solution of pair-correlation function (PCF):

$$\Theta(r) \equiv \langle \theta(\mathbf{r}+\mathbf{x})\theta(\mathbf{x})\rangle = C_2 \tau_p^2 \epsilon^{2/3} r^{-4/3}, \tag{1}$$

where C_2 is a universal constant, τ_p is relaxation timescale of relative velocity between particles and fluid, and ϵ is the mean dissipation rate of turbulence energy.

A closure theory for PCF further allows us to reach underlying physics behind Eq. (1); scale-local transfer of the number-density variance is discovered on the basis of dynamical equation of PCF, which guarantees the universal power law (1) free from both forcing-scale and dissipation-range properties. Our DNS on particle-laden homogeneous isotropic turbulence reasonably supports the predicted $-4/3$-power law of Eq. (1). Theoretical prediction $C_2 \approx 16 \pm 2$ well agrees with the DNS data.